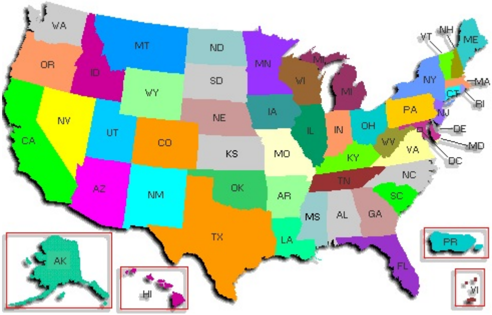
An Introduction to Coloring

Dan Swenson, Black Hills State University

February 4, 2013

Outline



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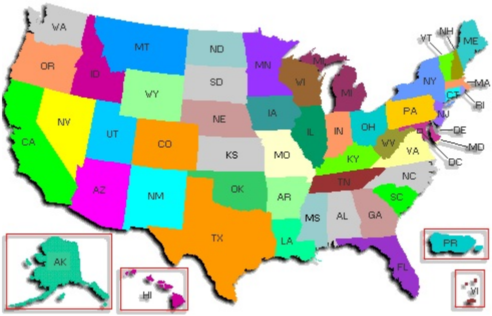

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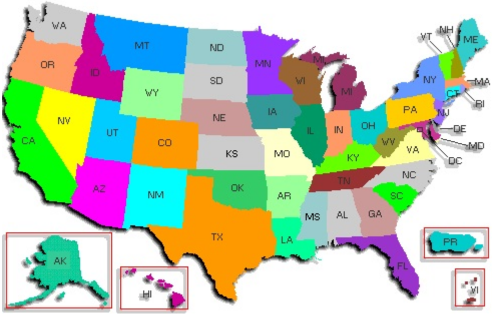

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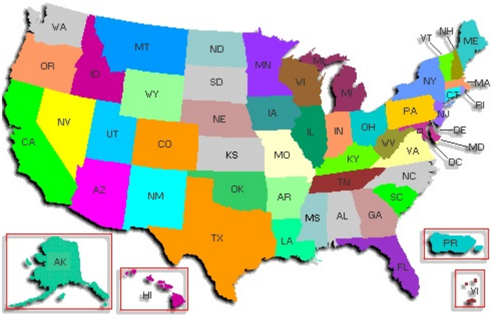

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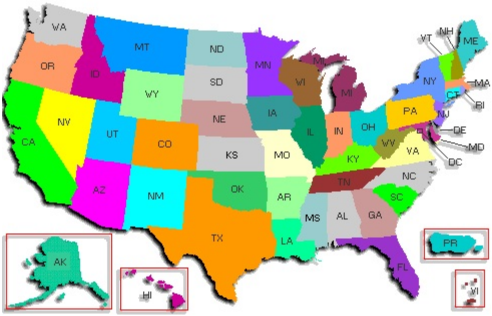

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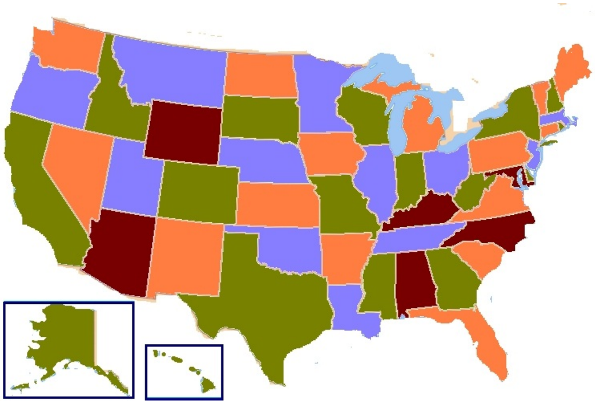
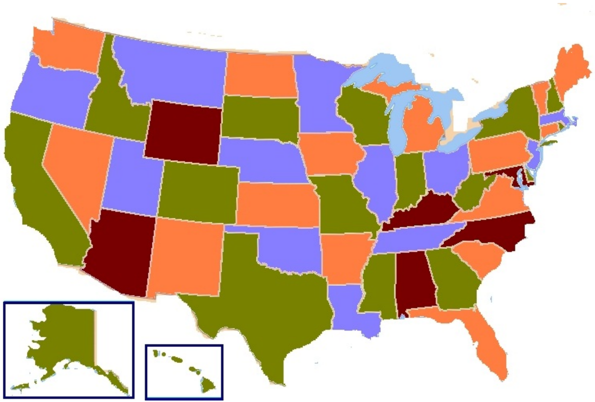


(from Wikipedia)

You can color the map with 4 colors (don't worry about the lakes).



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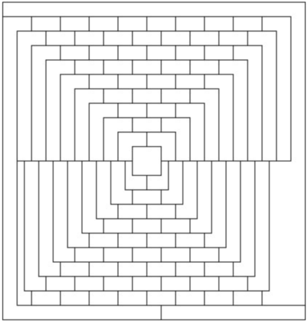

(-Martin Gardner, April, 1975)

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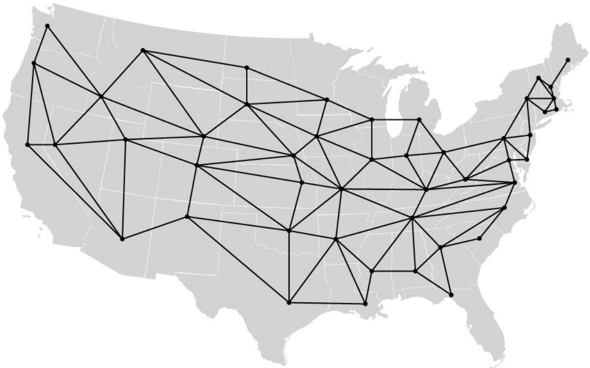

(-Randal Bryant)

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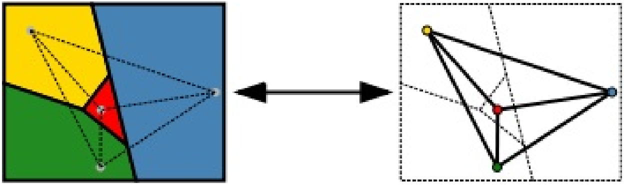

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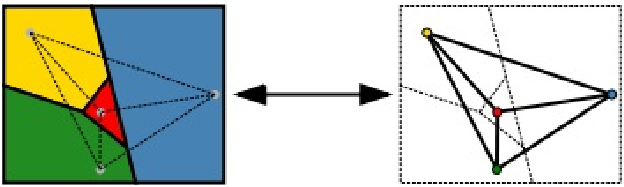

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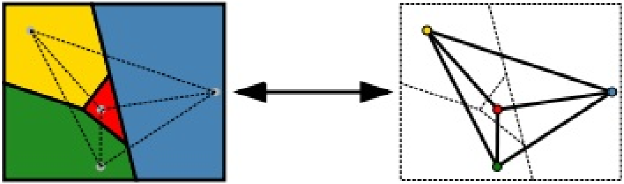

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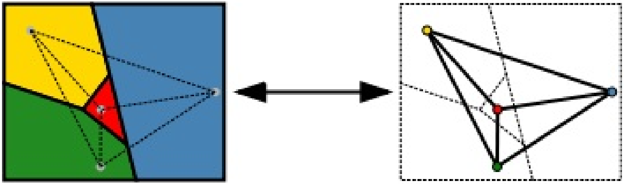

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A k-coloring of a graph is a coloring which uses k different colors.

The Four-Color Theorem

Theorem

Any planar graph (i.e., any graph that may be drawn in the plane, possibly with curved edges but without any edges crossing) may be colored using 4 (or possibly fewer) colors.

Map of states $\rightarrow$ planar graph $\rightarrow$ 4-coloring of graph $\rightarrow$ map coloring with 4 colors, so this statement solves the map coloring problem.

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Example: What is the least number of colors needed to properly color the vertices of the following graphs? (Note: the least number of colors required is called the "chromatic number" of the graph.)

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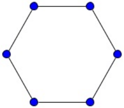

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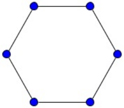

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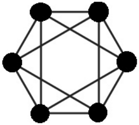

"Useful" example: Suppose we have 6 people (A, B, C, D, E, F) serving on 4 committees (1, 2, 3, 4):

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Thank you!

Other resources:

Wikipedia page on Four-Color Theorem
Interesting Map Problems. See also The McGregor Graph
Open Problems in Graph Theory